

Nearest neighbor recurrence relations for multiple orthogonal polynomials

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Abstract

We show that multiple orthogonal polynomials for r measures (μ_1, \dots, μ_r) satisfy a system of linear recurrence relations only involving nearest neighbor multi-indices $\vec{n} \pm \vec{e}_j$, where \vec{e}_j are the standard unit vectors. The recurrence coefficients are not arbitrary but satisfy a system of partial difference equations with boundary values given by the recurrence coefficients of the orthogonal polynomials with each of measures μ_j . We show how the Christoffel-Darboux formula for multiple orthogonal polynomials can be obtained easily using this information. We give explicit examples involving multiple Hermite, Charlier, Laguerre, and Jacobi polynomials.

1 Introduction

The three-term recurrence relation is an important piece of information when one is studying orthogonal polynomials. If μ is a positive measure on the real line for which all the moments

$$\nu_n = \int x^n d\mu(x)$$

exist, then we can define orthogonal polynomials $\{P_n, n = 0, 1, 2, \dots\}$ by

$$\int P_n(x) x^k d\mu(x) = 0, \quad k = 0, 1, \dots, n-1. \quad (1.1)$$

if we normalize the polynomials to be monic, i.e., $P_n(x) = x^n + \dots$, then these polynomials always exist and they are unique whenever the support of μ is an infinite set, and this is due to the fact that the Hankel matrix of moments

$$M_n = \begin{pmatrix} \nu_0 & \nu_1 & \cdots & \nu_n \\ \nu_1 & \nu_2 & \cdots & \nu_{n+1} \\ \vdots & \vdots & \cdots & \vdots \\ \nu_n & \nu_{n+1} & \cdots & \nu_{2n} \end{pmatrix}$$

is positive definite for all $n \geq 0$. The monic orthogonal polynomials are explicitly given by

$$P_n(x) = \frac{1}{\det M_{n-1}} \begin{vmatrix} \nu_0 & \nu_1 & \cdots & \nu_n \\ \nu_1 & \nu_2 & \cdots & \nu_{n+1} \\ \vdots & \vdots & \cdots & \vdots \\ \nu_{n-1} & \nu_n & \cdots & \nu_{2n-1} \\ 1 & x & \cdots & x^n \end{vmatrix}.$$

They satisfy a three-term recurrence relation

$$P_{n+1}(x) = (x - b_n)P_n(x) - a_n^2 P_{n-1}(x), \quad (1.2)$$

with initial values $P_0 = 1$ and $P_{-1} = 0$ and recurrence coefficients $b_n \in \mathbb{R}$ and $a_n^2 > 0$. An important result (the spectral theorem for orthogonal polynomials or Favard's theorem) is that any choice of recurrence coefficients $b_n \in \mathbb{R}$ and $a_{n+1}^2 > 0$ ($n = 0, 1, 2, \dots$) gives a sequence of monic orthogonal polynomials for some positive measure μ on the real line. This measure μ is the spectral measure of the Jacobi operator

$$J = \begin{pmatrix} b_0 & a_1 & 0 & 0 & 0 & 0 & \cdots \\ a_1 & b_1 & a_2 & 0 & 0 & 0 & \cdots \\ 0 & a_2 & b_2 & a_3 & 0 & 0 & \cdots \\ 0 & 0 & a_3 & \ddots & \ddots & & \\ \vdots & \vdots & & \ddots & & & \end{pmatrix}$$

or, when the operator is not self-adjoint, a self-adjoint extension of this operator. For the general theory of orthogonal polynomials we refer to Chihara [8], Freud [13], Gautschi [14], Ismail [15] and Szegő [19], and for the spectral theory we recommend Simon [18].

Recently a generalization of orthogonal polynomials has been considered. Let $\vec{n} = (n_1, n_2, \dots, n_r) \in \mathbb{N}^r$ be a multi-index of size $|\vec{n}| = n_1 + n_2 + \dots + n_r$ and suppose $\mu_1, \mu_2, \dots, \mu_r$ are positive measure on the real line. **Type I multiple orthogonals** are given by the vector $(A_{\vec{n},1}, \dots, A_{\vec{n},r})$, where $A_{\vec{n},j}$ is a polynomial of degree $\leq n_j - 1$, for which

$$\int x^k \sum_{j=1}^r A_{\vec{n},j}(x) w_j(x) d\mu(x) = 0, \quad k = 0, 1, \dots, |\vec{n}| - 2, \quad (1.3)$$

where $\mu = \mu_1 + \mu_2 + \dots + \mu_r$ and w_j is the Radon-Nikodym derivative $d\mu_j/d\mu$. We use the normalization

$$\int x^{|\vec{n}|-1} \sum_{j=1}^r A_{\vec{n},j}(x) w_j(x) d\mu(x) = 1. \quad (1.4)$$

The equations (1.3)–(1.4) are a linear system of $|\vec{n}|$ equations for the unknown coefficients of $A_{\vec{n},1}, \dots, A_{\vec{n},r}$. This system has a unique solution if the matrix of mixed moments

$$M_{\vec{n}} = \left(\begin{vmatrix} \nu_0^{(1)} & \nu_1^{(1)} & \cdots & \nu_{n_1-1}^{(1)} \\ \nu_1^{(1)} & \nu_2^{(1)} & \cdots & \nu_{n_1}^{(1)} \\ \vdots & \vdots & \cdots & \vdots \\ \nu_{|\vec{n}|-1}^{(1)} & \nu_{|\vec{n}|}^{(1)} & \cdots & \nu_{\vec{n}+n_1-2}^{(1)} \end{vmatrix} \cdots \cdots \begin{vmatrix} \nu_0^{(r)} & \nu_1^{(r)} & \cdots & \nu_{n_r-1}^{(r)} \\ \nu_1^{(r)} & \nu_2^{(r)} & \cdots & \nu_{n_r}^{(r)} \\ \vdots & \vdots & \cdots & \vdots \\ \nu_{|\vec{n}|-1}^{(r)} & \nu_{|\vec{n}|}^{(r)} & \cdots & \nu_{\vec{n}+n_r-2}^{(r)} \end{vmatrix} \right), \quad (1.5)$$

where

$$\nu_n^{(j)} = \int x^n d\mu_j(x), \quad j = 1, \dots, r,$$

is not singular, in which case we call the multi-index \vec{n} a *normal index*. The **type II multiple orthogonal polynomial** is the monic polynomial $P_{\vec{n}}(x) = x^{|\vec{n}|} + \dots$ of degree $|\vec{n}|$ for which

$$\begin{aligned} \int P_{\vec{n}}(x) x^k d\mu_1(x) &= 0, & k = 0, 1, \dots, n_1 - 1, \\ &\vdots \\ \int P_{\vec{n}}(x) x^k d\mu_r(x) &= 0, & k = 0, 1, \dots, n_r - 1. \end{aligned} \tag{1.6}$$

This gives a linear system of $|\vec{n}|$ equations for the $|\vec{n}|$ unknown coefficients of $P_{\vec{n}}$ and the matrix of this linear system is the transpose $M_{\vec{n}}^t$ of (1.5), hence $P_{\vec{n}}$ exists and is unique whenever \vec{n} is a normal index. See Aptekarev [2], Coussement and Van Assche [20], Nikishin and Sorokin [17, Chapter 4, §3], Ismail [15, Chapter 23]. The polynomials on the stepline, i.e.,

$$p_{kr+j} = P_{(k+1, \dots, k+1, k, \dots, k)}$$

are also known as d -orthogonal polynomials (with $d = r$), see Douak and Maroni [12], Ben Cheikh and Douak [5].

In this paper we assume that all multi-indices are normal and we investigate the nearest-neighbor recurrence relations

$$\begin{aligned} xP_{\vec{n}}(x) &= P_{\vec{n}+\vec{e}_1}(x) + b_{\vec{n},1}P_{\vec{n}}(x) + \sum_{j=1}^r a_{\vec{n},j}P_{\vec{n}-\vec{e}_j}(x), \\ &\vdots \\ xP_{\vec{n}}(x) &= P_{\vec{n}+\vec{e}_r}(x) + b_{\vec{n},r}P_{\vec{n}}(x) + \sum_{j=1}^r a_{\vec{n},j}P_{\vec{n}-\vec{e}_j}(x), \end{aligned} \tag{1.7}$$

where $\vec{e}_j = (0, \dots, 0, 1, 0, \dots, 0)$ is the j -th standard unit vector with 1 on the j -th entry, and $(a_{\vec{n},1}, \dots, a_{\vec{n},r})$ and $(b_{\vec{n},1}, \dots, b_{\vec{n},r})$ are the recurrence coefficients. These recurrence relations were derived in [15, Thm. 23.1.11] and it was shown that

$$a_{\vec{n},j} = \frac{\int x^{n_j} P_{\vec{n}}(x) d\mu_j(x)}{\int x^{n_j-1} P_{\vec{n}-\vec{e}_j}(x) d\mu_j(x)}, \tag{1.8}$$

and

$$b_{\vec{n},j} = \int xP_{\vec{n}}(x)Q_{\vec{n}+\vec{e}_j}(x) d\mu(x), \tag{1.9}$$

where $Q_{\vec{n}} = \sum_{j=1}^r A_{\vec{n},j}w_j$. There are similar recurrence relations for the type I multiple

orthogonal polynomials:

$$\begin{aligned}
xQ_{\vec{n}}(x) &= Q_{\vec{n}-\vec{e}_1}(x) + b_{\vec{n}-\vec{e}_1,1}Q_{\vec{n}}(x) + \sum_{j=1}^r a_{\vec{n},j}Q_{\vec{n}+\vec{e}_j}(x), \\
&\vdots \\
xQ_{\vec{n}}(x) &= Q_{\vec{n}-\vec{e}_r}(x) + b_{\vec{n}-\vec{e}_r,r}Q_{\vec{n}}(x) + \sum_{j=1}^r a_{\vec{n},j}Q_{\vec{n}+\vec{e}_j}(x).
\end{aligned} \tag{1.10}$$

Observe that the same recurrence coefficients $(a_{\vec{n},1}, \dots, a_{\vec{n},r})$ are used but that there is a shift in the other recurrence coefficients $(b_{\vec{n}-\vec{e}_j,j})_{1 \leq j \leq r}$.

In Section 2 we will show how to find these recurrence relations from a Riemann-Hilbert problem for multiple orthogonal polynomials, which was given by Van Assche, Geronimo and Kuijlaars [21]. This approach is not new but merely serves to show that these nearest neighbor recurrence relations enter naturally. In Section 3 it will be shown that not every choice of recurrence coefficients $(a_{\vec{n},1}, \dots, a_{\vec{n},r})$ and $(b_{\vec{n},1}, \dots, b_{\vec{n},r})$ is possible: our main result is that the recurrence coefficients satisfy a system of partial difference equations. The boundary conditions are given by the recurrence coefficients of the orthogonal polynomials with measure μ_j whenever $\vec{n} = n\vec{e}_j$, i.e., when all the components in the multi-index \vec{n} are zero except for the j -th component. As such the recurrence coefficients of multiple orthogonal polynomials are the solution of a discrete integrable system on the lattice \mathbb{N}^r . We will first work with the case $r = 2$ to simplify the notation and then give the general result for $r \geq 2$. In Section 4 we will show how the nearest neighbor recurrence relations and the partial difference equations for the recurrence coefficients can be used to give a simple proof of the Christoffel-Darboux formula for multiple orthogonal polynomials, which was first given by Daems and Kuijlaars [10]; see [1] for a Christoffel-Darboux formula for multiple orthogonal polynomials of mixed type. This Christoffel-Darboux formula plays an important role in the analysis of certain random matrices [6] [16] and non-intersecting Brownian motions [11]. In Section 5 we will give several examples of multiple orthogonal polynomials and give their recurrence coefficients explicitly. For some of these examples the recurrence coefficients are given here for the first time.

2 The Riemann-Hilbert problem

In this section we will take $r = 2$ to keep the notation simple. The extension to general r is straightforward but requires bigger matrices. Suppose that the measures μ_1 and μ_2 are absolutely continuous with weights w_1 and w_2 . If w_1 and w_2 are Hölder continuous, then we can formulate the following Riemann-Hilbert problem: find a 3×3 matrix function Y such that

1. Y is analytic on $\mathbb{C} \setminus \mathbb{R}$,
2. the limits $Y_{\pm}(x) = \lim_{\epsilon \rightarrow 0+} Y(x \pm i\epsilon)$ exist and

$$Y_+(x) = Y_-(x) \begin{pmatrix} 1 & w_1(x) & w_2(x) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x \in \mathbb{R}, \tag{2.1}$$

3. For $z \rightarrow \infty$ one has

$$Y(z) = (I + \mathcal{O}(1/z)) \begin{pmatrix} z^{n+m} & 0 & 0 \\ 0 & z^{-n} & 0 \\ 0 & 0 & z^{-m} \end{pmatrix}. \quad (2.2)$$

In case w_1 or w_2 are defined on bounded intervals or semi-infinite intervals one additionally needs to specify the behavior of Y near the endpoints of the intervals. In [21] it was shown that this Riemann-Hilbert problem has a unique solution in terms of type II multiple orthogonal polynomials when (n, m) , $(n-1, m)$ and $(n, m-1)$ are normal indices, i.e.,

$$Y = \begin{pmatrix} P_{n,m} & C(P_{n,m}w_1) & C(P_{n,m}w_2) \\ c_1(n, m)P_{n-1,m} & c_1C(P_{n-1,m}w_1) & c_1C(P_{n-1,m}w_2) \\ c_2(n, m)P_{n,m-1} & c_2C(P_{n,m-1}w_1) & c_2C(P_{n,m-1}w_2) \end{pmatrix} \quad (2.3)$$

where the Cauchy transform is used

$$C(Pw) = \frac{1}{2\pi i} \int \frac{P(x)w(x)}{x-z} dx$$

and the constants c_1 and c_2 are given by

$$\begin{aligned} \frac{1}{c_1(n, m)} &= \frac{-1}{2\pi i} \int P_{n-1,m}(x)x^{n-1}w_1(x) dx, \\ \frac{1}{c_2(n, m)} &= \frac{-1}{2\pi i} \int P_{n,m-1}(x)x^{m-1}w_2(x) dx. \end{aligned}$$

In case μ_1 and μ_2 are discrete measures on the set $A = \{x_1, x_2, x_3, \dots\}$ one can formulate a Riemann-Hilbert problem for a meromorphic matrix function: find a 3×3 matrix function Y such that:

1. Y is meromorphic on \mathbb{C} with poles of order one at the points in A ,
2. the residue of Y at x_k is given by

$$\lim_{z \rightarrow x_k} Y(z) \begin{pmatrix} 0 & \mu_1(\{x_k\}) & \mu_2(\{x_k\}) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

3. Y has the asymptotic condition (2.2).

The solution of this problem is

$$Y(z) = \begin{pmatrix} P_{n,m}(z) & \sum_{k=1}^{\infty} \frac{P_{n,m}(x_k)}{z-x_k} \mu_1(\{x_k\}) & \sum_{k=1}^{\infty} \frac{P_{n,m}(x_k)}{z-x_k} \mu_2(\{x_k\}) \\ \hat{c}_1(n, m)P_{n-1,m}(z) & \hat{c}_1 \sum_{k=1}^{\infty} \frac{P_{n-1,m}(x_k)}{z-x_k} \mu_1(\{x_k\}) & \hat{c}_1 \sum_{k=1}^{\infty} \frac{P_{n-1,m}(x_k)}{z-x_k} \mu_2(\{x_k\}) \\ \hat{c}_2(n, m)P_{n,m-1}(z) & \hat{c}_2 \sum_{k=1}^{\infty} \frac{P_{n,m-1}(x_k)}{z-x_k} \mu_1(\{x_k\}) & \hat{c}_2 \sum_{k=1}^{\infty} \frac{P_{n,m-1}(x_k)}{z-x_k} \mu_2(\{x_k\}) \end{pmatrix}$$

where

$$\begin{aligned} \frac{1}{\hat{c}_1(n, m)} &= \sum_{k=1}^{\infty} P_{n-1,m}(x_k)x_k^{n-1}\mu_1(\{x_k\}) \\ \frac{1}{\hat{c}_2(n, m)} &= \sum_{k=1}^{\infty} P_{n,m-1}(x_k)x_k^{m-1}\mu_2(\{x_k\}). \end{aligned}$$

We will derive the recurrence relations for type II multiple orthogonal polynomials from the Riemann-Hilbert problem. The result is not limited to Hölder continuous weights w_1 and w_2 but holds in general and can be derived from the biorthogonality [15, Thm. 23.1.6]

$$\int P_{\vec{n}}(x)Q_{\vec{m}}(w) d\mu(x) = \begin{cases} 0, & \text{if } \vec{m} \leq \vec{n}, \\ 0, & \text{if } |\vec{n}| \leq |\vec{m}| - 2, \\ 1, & \text{if } |\vec{m}| = |\vec{n}| + 1. \end{cases}$$

We merely use the Riemann-Hilbert problem to indicate that the nearest-neighbor recurrence relations come out in a natural way.

Theorem 2.1. *Suppose all multi-indices $(n, m) \in \mathbb{N}^2$ are normal. Then the type II multiple orthogonal polynomials satisfy the system of recurrence relations*

$$P_{n+1,m}(x) = (x - c_{n,m})P_{n,m}(x) - a_{n,m}P_{n-1,m}(x) - b_{n,m}P_{n,m-1}(x), \quad (2.4)$$

$$P_{n,m+1}(x) = (x - d_{n,m})P_{n,m}(x) - a_{n,m}P_{n-1,m}(x) - b_{n,m}P_{n,m-1}(x), \quad (2.5)$$

with $a_{0,m} = 0$ and $b_{n,0} = 0$ for all $n, m \geq 0$.

Proof. First of all we observe that $\det Y$ is an analytic function in $\mathbb{C} \setminus \mathbb{R}$ which has no jump on the real axis, hence $\det Y$ is an entire function. Its behavior near infinity is $\det Y(z) = 1 + \mathcal{O}(1/z)$ hence by Liouville's theorem we find that $\det Y = 1$. We can therefore consider the matrix

$$R_1(n, m) = Y_{n+1,m}Y_{n,m}^{-1},$$

where the subscript (n, m) is used for the Riemann-Hilbert problem with the type II multiple orthogonal polynomial $P_{n,m}$ in the entry of the first row and first column of $Y_{n,m}$. Clearly R_1 is an analytic function on $\mathbb{C} \setminus \mathbb{R}$, and since $Y_{n,m}$ and $Y_{n+1,m}$ have the same jump matrix on \mathbb{R} we see that R_1 has no jump on \mathbb{R} . Hence R_1 is an entire matrix function. If we write the asymptotic condition (2.2) as

$$Y_{n,m}(z) = \left(I + \frac{A(n, m)}{z} + \mathcal{O}(1/z^2) \right) \begin{pmatrix} z^{n+m} & 0 & 0 \\ 0 & z^{-n} & 0 \\ 0 & 0 & z^{-m} \end{pmatrix},$$

where $A(n, m)$ is the 3×3 matrix coefficient of $1/z$ in the $\mathcal{O}(1/z)$ term of (2.2), then after some calculus we find

$$R_1(n, m) = \begin{pmatrix} z + A_{1,1}(n+1, m) - A_{1,1}(n, m) & -A_{1,2}(n, m) & -A_{1,3}(n, m) \\ A_{2,1}(n+1, m) & 0 & 0 \\ A_{3,1}(n+1, m) & 0 & 1 \end{pmatrix} + \mathcal{O}(1/z),$$

where $A_{i,j}(n, m)$ is the entry on row i and column j of $A(n, m)$. Liouville's theorem then implies that R_1 is the matrix polynomial

$$R_1(n, m) = \begin{pmatrix} z + A_{1,1}(n+1, m) - A_{1,1}(n, m) & -A_{1,2}(n, m) & -A_{1,3}(n, m) \\ A_{2,1}(n+1, m) & 0 & 0 \\ A_{3,1}(n+1, m) & 0 & 1 \end{pmatrix}, \quad (2.6)$$

and we can therefore write

$$Y_{n+1,m} = R_1(n, m)Y_{n,m}. \quad (2.7)$$

In a similar way we also have

$$Y_{n,m+1} = R_2(n, m)Y_{n,m}, \quad (2.8)$$

with

$$R_2(n, m) = \begin{pmatrix} z + A_{1,1}(n, m+1) - A_{1,1}(n, m) & -A_{1,2}(n, m) & -A_{1,3}(n, m) \\ A_{2,1}(n, m+1) & 1 & 0 \\ A_{3,1}(n, m+1) & 0 & 0 \end{pmatrix}. \quad (2.9)$$

If we now work out the $(1, 1)$ -entry of (2.7), then we find (2.4) with

$$c_{n,m} = A_{1,1}(n, m) - A_{1,1}(n+1, m)$$

and

$$a_{n,m} = c_1(n, m)A_{1,2}(n, m), \quad b_{n,m} = c_2(n, m)A_{1,3}(n, m).$$

Similarly, if we work out the $(1, 1)$ -entry of (2.8), then we find (2.5) with

$$d_{n,m} = A_{1,1}(n, m) - A_{1,1}(n, m+1).$$

□

Observe that $\det R_1(n, m) = 1$ which implies that $A_{1,2}(n, m)A_{2,1}(n+1, m) = 1$. If we work out the $(2, 1)$ -entry of (2.7) then we find

$$c_1(n+1, m)P_{n,m} = A_{2,1}(n+1, m)P_{n,m},$$

so that $c_1(n+1, m) = A_{2,1}(n+1, m)$. Combined with the previous result, this gives

$$a_{n,m} = \frac{c_1(n, m)}{c_1(n+1, m)},$$

which corresponds to (1.8) when $r = 2$. Similarly the identity $\det R_2(n, m) = 1$ implies that $A_{1,3}(n, m)A_{3,1}(n, m+1) = 1$ and the $(3, 1)$ -entry of (2.8) gives $c_2(n, m+1) = A_{3,1}(n, m+1)$, so that

$$b_{n,m} = \frac{c_2(n, m)}{c_2(n, m+1)}.$$

Theorem 2.2. *Denote the moments by*

$$\nu_k^{(j)} = \int x^k w_j(x) dx$$

and the moment matrix by

$$M_{n,m} = \begin{pmatrix} \nu_0^{(1)} & \nu_1^{(1)} & \cdots & \nu_{n-1}^{(1)} & \nu_0^{(2)} & \nu_1^{(2)} & \cdots & \nu_{m-1}^{(2)} \\ \nu_1^{(1)} & \nu_2^{(1)} & \cdots & \nu_n^{(1)} & \nu_1^{(2)} & \nu_2^{(2)} & \cdots & \nu_m^{(2)} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \nu_{n+m-1}^{(1)} & \nu_{n+m}^{(1)} & \cdots & \nu_{2n+m-2}^{(1)} & \nu_{n+m-1}^{(2)} & \nu_{n+m}^{(2)} & \cdots & \nu_{n+2m-2}^{(2)} \end{pmatrix}. \quad (2.10)$$

Then the recurrence coefficients can be written as

$$a_{n,m} = \frac{\det M_{n+1,m} \det M_{n-1,m}}{(\det M_{n,m})^2}.$$

and

$$b_{n,m} = \frac{\det M_{n,m+1} \det M_{n,m-1}}{(\det M_{n,m})^2}.$$

Furthermore we have

$$d_{n,m} - c_{n,m} = \frac{\det M_{n,m} \det M_{n+1,m+1}}{\det M_{n+1,m} \det M_{n,m+1}}.$$

Proof. The type II multiple orthogonal polynomial can be written as

$$P_{n,m}(x) = \frac{1}{\det M_{n,m}} \begin{vmatrix} \nu_0^{(1)} & \nu_1^{(1)} & \cdots & \nu_{n-1}^{(1)} & \nu_0^{(2)} & \nu_1^{(2)} & \cdots & \nu_{m-1}^{(2)} & 1 \\ \nu_1^{(1)} & \nu_2^{(1)} & \cdots & \nu_n^{(1)} & \nu_1^{(2)} & \nu_2^{(2)} & \cdots & \nu_m^{(2)} & x \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \nu_{n+m}^{(1)} & \nu_{n+m+1}^{(1)} & \cdots & \nu_{2n+m-1}^{(1)} & \nu_{n+m}^{(2)} & \nu_{n+m+1}^{(2)} & \cdots & \nu_{n+2m-1}^{(2)} & x^{n+m} \end{vmatrix},$$

where $M_{n,m}$ is the moment matrix given in (2.10). This formula indeed has all the orthogonality properties (1.6), and from it one easily finds

$$\int x^n P_{n,m}(x) w_1(x) dx = \frac{(-1)^m \det M_{n+1,m}}{\det M_{n,m}},$$

so that

$$a_{n,m} = \frac{\det M_{n+1,m} \det M_{n-1,m}}{(\det M_{n,m})^2}.$$

In a similar way

$$\int x^m P_{n,m}(x) w_2(x) dx = \frac{\det M_{n,m+1}}{\det M_{n,m}}$$

and

$$b_{n,m} = \frac{\det M_{n,m+1} \det M_{n,m-1}}{(\det M_{n,m})^2}.$$

Take the $(3, 1)$ -entry of (2.7) to find

$$c_2(n+1, m) P_{n+1,m-1} = A_{3,1}(n+1, m) P_{n,m} + c_2(n, m) P_{n,m-1}.$$

On the other hand, if we subtract (2.4) from (2.5) then we have

$$P_{n,m+1} - P_{n+1,m} = (c_{n,m} - d_{n,m}) P_{n,m}.$$

If we increase m by one in the penultimate formula, then comparison with the previous formula gives

$$d_{n,m} - c_{n,m} = \frac{c_2(n, m+1)}{c_2(n+1, m+1)} = \frac{\det M_{n,m} \det M_{n+1,m+1}}{\det M_{n+1,m} \det M_{n,m+1}}. \quad (2.11)$$

Similarly, if we take the $(2, 1)$ -entry of (2.8), then

$$c_1(n, m+1)P_{n-1, m+1} = A_{2,1}(n, m+1)P_{n, m} + c_1(n, m)P_{n-1, m},$$

and if we increase n by one and compare with the previous formulas, then

$$d_{n, m} - c_{n, m} = -\frac{c_1(n+1, m)}{c_1(n+1, m+1)} = \frac{\det M_{n, m} \det M_{n+1, m+1}}{\det M_{n+1, m} \det M_{n, m+1}}. \quad (2.12)$$

□

Observe that the arrays c and d can be obtained if one knows the array $A_{1,1}$ and the latter contains the second coefficients of the type II multiple orthogonal polynomials

$$P_{n, m}(x) = x^{n+m} + A_{1,1}(n, m)x^{n+m-1} + \dots$$

Each $A_{1,1}(n, m)$ can also be expressed as the ratio of two moment matrices,

$$A_{1,1}(n, m) = -\frac{\det \widehat{M}_{n, m}}{\det M_{n, m}}$$

and the moment matrix in the numerator is

$$\widehat{M}_{n, m} \begin{pmatrix} \nu_0^{(1)} & \nu_1^{(1)} & \cdots & \nu_{n-1}^{(1)} & \nu_0^{(2)} & \nu_1^{(2)} & \cdots & \nu_{m-1}^{(2)} \\ \nu_1^{(1)} & \nu_2^{(1)} & \cdots & \nu_n^{(1)} & \nu_1^{(2)} & \nu_2^{(2)} & \cdots & \nu_m^{(2)} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \nu_{n+m-2}^{(1)} & \nu_{n+m-1}^{(1)} & \cdots & \nu_{2n+m-3}^{(1)} & \nu_{n+m-2}^{(2)} & \nu_{n+m-1}^{(2)} & \cdots & \nu_{n+2m-3}^{(2)} \\ \nu_{n+m}^{(1)} & \nu_{n+m+1}^{(1)} & \cdots & \nu_{2n+m-1}^{(1)} & \nu_{n+m}^{(2)} & \nu_{n+m+1}^{(2)} & \cdots & \nu_{n+2m-1}^{(2)} \end{pmatrix},$$

i.e., it is the moment matrix $M_{n, m}$ from (2.10) but with the last row replaced by

$$\nu_{n+m}^{(1)} \quad \nu_{n+m+1}^{(1)} \quad \cdots \quad \nu_{2n+m-1}^{(1)} \quad \nu_{n+m}^{(2)} \quad \nu_{n+m+1}^{(2)} \quad \cdots \quad \nu_{n+2m-1}^{(2)}$$

3 Partial difference equations

We will first consider the case $r = 2$ to simplify the notation. The case for general r is handled in the second part of the section.

The recurrence coefficients $a_{n, m}, b_{n, m}, c_{n, m}, d_{n, m}$ are not arbitrary sequences. They satisfy certain partial difference equations, which are given in the following theorem.

Theorem 3.1. *Suppose that all the indices $(n, m) \in \mathbb{N}^2$ are normal. The recurrence coefficients in the recurrence relations (2.4)–(2.5) for type II multiple orthogonal polynomials satisfy the following equations*

$$d_{n+1, m} - d_{n, m} = c_{n, m+1} - c_{n, m}, \quad (3.1)$$

$$b_{n+1, m} - b_{n, m+1} + a_{n+1, m} - a_{n, m+1} = \det \begin{pmatrix} d_{n+1, m} & d_{n, m} \\ c_{n, m+1} & c_{n, m} \end{pmatrix}, \quad (3.2)$$

$$\frac{a_{n, m+1}}{a_{n, m}} = \frac{c_{n, m} - d_{n, m}}{c_{n-1, m} - d_{n-1, m}}, \quad (3.3)$$

$$\frac{b_{n+1, m}}{b_{n, m}} = \frac{c_{n, m} - d_{n, m}}{c_{n, m-1} - d_{n, m-1}}. \quad (3.4)$$

Proof. We obtain these equations by observing that $Y_{n+1,m+1}$ can be obtained in two different ways from the information in $Y_{n,m}$. First we can compute $Y_{n+1,m}$ by using (2.4) and then we can use (2.5) (with n replaced by $n+1$) to find $Y_{n+1,m+1}$, i.e.

$$Y_{n+1,m+1} = R_2(n+1, m)R_1(n, m)Y_{n,m}.$$

On the other hand we can also compute $Y_{n,m+1}$ by using (2.5) and then use (2.4) (with m replaced by $m+1$) to find $Y_{n+1,m+1}$,

$$Y_{n+1,m+1} = R_1(n, m+1)R_2(n, m)Y_{n,m}.$$

Since $\det Y = 1$ this implies that

$$R_2(n+1, m)R_1(n, m) = R_1(n, m+1)R_2(n, m). \quad (3.5)$$

Observe that

$$R_1(n, m) = \begin{pmatrix} z - c_{n,m} & -a_{n,m}/c_1(n, m) & -b_{n,m}/c_2(n, m) \\ c_1(n+1, m) & 0 & 0 \\ c_2(n+1, m) & 0 & 1 \end{pmatrix},$$

$$R_2(n, m) = \begin{pmatrix} z - d_{n,m} & -a_{n,m}/c_1(n, m) & -b_{n,m}/c_2(n, m) \\ c_1(n, m+1) & 1 & 0 \\ c_2(n, m+1) & 0 & 0 \end{pmatrix},$$

hence (3.5) is equivalent to

$$\begin{pmatrix} S_{1,1}(z) & S_{1,2}(z) & S_{1,3}(z) \\ S_{2,1}(z) & 0 & 0 \\ S_{3,1}(z) & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where

$$\begin{aligned} S_{1,1}(z) &= (z - c_{n,m})(z - d_{n+1,m}) - a_{n+1,m} - b_{n+1,m} - (z - d_{n,m})(z - c_{n,m+1}) \\ &\quad + a_{n,m+1} + b_{n,m+1}, \\ S_{1,2}(z) &= -(z - d_{n+1,m})\frac{a_{n,m}}{c_1(n, m)} + (z - c_{n,m+1})\frac{a_{n,m}}{c_1(n, m)} + \frac{a_{n,m+1}}{c_1(n, m+1)} \\ S_{1,3}(z) &= -(z - d_{n+1,m})\frac{b_{n,m}}{c_2(n, m)} - \frac{b_{n+1,m}}{c_2(n+1, m)} + (z - c_{n,m+1})\frac{b_{n,m}}{c_2(n, m)} \\ S_{2,1}(z) &= (z - c_{n,m})c_1(n+1, m+1) + c_1(n+1, m) - (z - d_{n,m})c_1(n+1, m+1) \\ S_{3,1}(z) &= (z - c_{n,m})c_2(n+1, m+1) - (z - d_{n,m})c_2(n+1, m+1) - c_2(n, m+1). \end{aligned}$$

Observe that $S_{1,1}$ is a polynomial of degree one and the remaining quantities $S_{1,2}$, $S_{1,3}$, $S_{2,1}$ and $S_{3,1}$ are polynomials of degree zero. If we set the coefficient of z in $S_{1,1}$ equal to zero, then we find (3.1). If we put the constant term in $S_{1,1}$ equal to zero, then we find (3.2). If we put $S_{1,2}$ equal to zero and use (2.12), then we find (3.3), and if we put $S_{1,3}$ equal to zero and use (2.11), then we find (3.4). If we put $S_{2,1}$ and $S_{3,1}$ equal to zero, then we find (2.12) and (2.11). \square

If we introduce the difference operators $\Delta_1, \Delta_2, \nabla_1$ and ∇_2 for which

$$\Delta_1 f(n, m) = f(n+1, m) - f(n, m), \quad \Delta_2 f(n, m) = f(n, m+1) - f(n, m)$$

$$\nabla_1 f(n, m) = f(n, m) - f(n-1, m), \quad \nabla_2 f(n, m) = f(n, m) - f(n, m-1),$$

then we find the partial difference equations

$$\begin{aligned} \Delta_1 d &= \Delta_2 c, \\ \Delta_1(a+b) - \Delta_2(a+b) &= \det \begin{pmatrix} \Delta_1 d & d \\ \Delta_2 c & c \end{pmatrix} \\ \Delta_2 \log a = \nabla_1 \log(c-d), & \quad \Delta_1 \log b = \nabla_2 \log(c-d). \end{aligned}$$

The result for general r has more partial difference relations. In fact for each pair (i, j) with $i \neq j$ we have partial difference relations as in Theorem 3.1 for the partial difference operators Δ_i, Δ_j and ∇_i, ∇_j with

$$\Delta_i f(\vec{n}) = f(\vec{n} + \vec{e}_i) - f(\vec{n}), \quad \nabla_i f(\vec{n}) = f(\vec{n}) - f(\vec{n} - \vec{e}_i).$$

We will not use the Riemann-Hilbert problem explicitly but some of the elements of the proof use ideas from the Riemann-Hilbert approach, in particular the transfer matrices $R_k(\vec{n})$ are inspired by the matrices $R_1(n, m)$ and $R_2(n, m)$ which we used for the case $r = 2$.

Theorem 3.2. *Suppose all multi-indices $\vec{n} \in \mathbb{N}^r$ are normal. Suppose $1 \leq i \neq j \leq r$, then the recurrence coefficients for the nearest neighbor recurrence relations (1.7) satisfy*

$$b_{\vec{n}+\vec{e}_i, j} - b_{\vec{n}, j} = b_{\vec{n}+\vec{e}_j, i} - b_{\vec{n}, i}, \quad (3.6)$$

$$\sum_{k=1}^r a_{\vec{n}+\vec{e}_j, k} - \sum_{k=1}^r a_{\vec{n}+\vec{e}_i, k} = \det \begin{pmatrix} b_{\vec{n}+\vec{e}_j, i} & b_{\vec{n}, i} \\ b_{\vec{n}+\vec{e}_i, j} & b_{\vec{n}, j} \end{pmatrix}, \quad (3.7)$$

$$\frac{a_{\vec{n}, i}}{a_{\vec{n}+\vec{e}_j, i}} = \frac{b_{\vec{n}-\vec{e}_i, j} - b_{\vec{n}-\vec{e}_i, i}}{b_{\vec{n}, j} - b_{\vec{n}, i}}, \quad (3.8)$$

Proof. The recurrence relations (1.7) give a remarkable relation between three neighboring polynomials:

$$P_{\vec{n}+\vec{e}_k}(x) - P_{\vec{n}+\vec{e}_j}(x) = (b_{\vec{n}, j} - b_{\vec{n}, k})P_{\vec{n}}(x).$$

This relation and the k -th relation in (1.7) can be written as

$$\begin{pmatrix} P_{\vec{n}+\vec{e}_k}(x) \\ P_{\vec{n}+\vec{e}_k-\vec{e}_1}(x) \\ P_{\vec{n}+\vec{e}_k-\vec{e}_2}(x) \\ \vdots \\ P_{\vec{n}}(x) \\ \vdots \\ P_{\vec{n}+\vec{e}_k-\vec{e}_r}(x) \end{pmatrix} = \begin{pmatrix} x - b_{\vec{n}, k} & -a_{\vec{n}, 1} & -a_{\vec{n}, 2} & \cdots & -a_{\vec{n}, k} & \cdots & -a_{\vec{n}, r} \\ 1 & B_{k, 1}(\vec{n}) & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & B_{k, 2}(\vec{n}) & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & 0 & 0 & B_{k, r}(\vec{n}) \end{pmatrix} \begin{pmatrix} P_{\vec{n}}(x) \\ P_{\vec{n}-\vec{e}_1}(x) \\ P_{\vec{n}-\vec{e}_2}(x) \\ \vdots \\ P_{\vec{n}-\vec{e}_k}(x) \\ \vdots \\ P_{\vec{n}-\vec{e}_r}(x) \end{pmatrix}.$$

where $B_{k, j}(\vec{n}) = b_{\vec{n}-\vec{e}_j, j} - b_{\vec{n}-\vec{e}_j, k}$ (note that $B_{k, k}(\vec{n}) = 0$). We will write this matrix relation as

$$Y_{\vec{n}+\vec{e}_k} = R_k(\vec{n})Y_{\vec{n}}.$$

Note that $Y_{\vec{n}}$ is now a vector containing the multiple orthogonal polynomial $P_{\vec{n}}$ and its neighbors from below, and not a matrix as in the Riemann-Hilbert problem. Now we observe that there are two ways to obtain $Y_{\vec{n}+\vec{e}_i+\vec{e}_j}$ when $i \neq j$. The first way is to first compute $Y_{\vec{n}+\vec{e}_i}$ from $Y_{\vec{n}}$ and then to compute $Y_{\vec{n}+\vec{e}_i+\vec{e}_j}$ from $Y_{\vec{n}+\vec{e}_i}$:

$$Y_{\vec{n}+\vec{e}_i+\vec{e}_j} = R_j(\vec{n} + \vec{e}_i)R_i(\vec{n})Y_{\vec{n}}.$$

The second way is to first compute $Y_{\vec{n}+\vec{e}_j}$ from $Y_{\vec{n}}$ and then to compute $Y_{\vec{n}+\vec{e}_i+\vec{e}_j}$ from $Y_{\vec{n}+\vec{e}_j}$:

$$Y_{\vec{n}+\vec{e}_i+\vec{e}_j} = R_i(\vec{n} + \vec{e}_j)R_j(\vec{n})Y_{\vec{n}}.$$

The compatibility between these two ways implies that

$$R_j(\vec{n} + \vec{e}_i)R_i(\vec{n})Y_{\vec{n}} = R_i(\vec{n} + \vec{e}_j)R_j(\vec{n})Y_{\vec{n}}.$$

The polynomials in the vector $Y_{\vec{n}}$ are linearly independent whenever \vec{n} is a normal index. Indeed, if (c_0, c_1, \dots, c_r) are such that

$$c_0 P_{\vec{n}} + \sum_{j=1}^r c_j P_{\vec{n}-\vec{e}_j} = 0,$$

then surely $c_0 = 0$ follows by comparing the leading coefficients. If we multiply by x^{n_k-1} and integrate with respect to μ_k , then

$$c_k \int x^{n_k-1} P_{\vec{n}-\vec{e}_k}(x) d\mu_k(x) = 0$$

so that $c_k = 0$ follows, since the integral does not vanish. Indeed, if the integral vanishes, then $P_{\vec{n}} - aP_{\vec{n}-\vec{e}_k}$ is (for every a) a monic polynomial of degree $|\vec{n}|$ satisfying the orthogonality relations (1.6), which contradicts the normality of \vec{n} . The linear independence of the polynomials in $Y_{\vec{n}}$ implies that

$$R_j(\vec{n} + \vec{e}_i)R_i(\vec{n}) = R_i(\vec{n} + \vec{e}_j)R_j(\vec{n}). \quad (3.9)$$

The transfer matrix $R_k(\vec{n})$ can be written as

$$R_k(\vec{n}) = \begin{pmatrix} x - b_{\vec{n},k} & -\vec{a}_{\vec{n}}^t \\ \vec{1} & B_k(\vec{n}) \end{pmatrix}$$

where $\vec{a}_{\vec{n}}$ and $\vec{1}$ are (column) vectors of length r containing the $a_{\vec{n},j}$ ($1 \leq j \leq r$) and r ones respectively, and $B_k(\vec{n})$ is a diagonal matrix of size r containing the $B_{k,j}(\vec{n})$ ($1 \leq j \leq r$) on the diagonal. We then have

$$\begin{aligned} & R_i(\vec{n} + \vec{e}_j)R_j(\vec{n}) \\ &= \begin{pmatrix} (x - b_{\vec{n}+\vec{e}_j,i})(x - b_{\vec{n},j}) - \sum_{k=1}^r a_{\vec{n}+\vec{e}_j,k} & -(x - b_{\vec{n}+\vec{e}_j,i})\vec{a}_{\vec{n}}^t - \vec{a}_{\vec{n}+\vec{e}_j}^t B_j(\vec{n}) \\ (x - b_{\vec{n},j})\vec{1} + B_i(\vec{n} + \vec{e}_j)\vec{1} & -\vec{1} \cdot \vec{a}_{\vec{n}}^t + B_i(\vec{n} + \vec{e}_j)B_j(\vec{n}) \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} & R_j(\vec{n} + \vec{e}_i)R_i(\vec{n}) \\ &= \begin{pmatrix} (x - b_{\vec{n}+\vec{e}_i,j})(x - b_{\vec{n},i}) - \sum_{k=1}^r a_{\vec{n}+\vec{e}_i,k} & -(x - b_{\vec{n}+\vec{e}_i,j})\vec{a}_{\vec{n}}^t - \vec{a}_{\vec{n}+\vec{e}_i}^t B_i(\vec{n}) \\ (x - b_{\vec{n},i})\vec{1} + B_j(\vec{n} + \vec{e}_i)\vec{1} & -\vec{1} \cdot \vec{a}_{\vec{n}}^t + B_j(\vec{n} + \vec{e}_i)B_i(\vec{n}) \end{pmatrix}. \end{aligned}$$

If we compare the $(1, 1)$ -entries in (3.9) then this gives a polynomial of degree 1 which is identically zero. The coefficient of x vanishes whenever (3.6) holds and the constant term vanishes whenever (3.7) holds. If we check the remaining entries on the first row of (3.9), then we have for $1 \leq k \leq r$

$$b_{\vec{n}+\vec{e}_i,j}a_{\vec{n},k} - B_{i,k}(\vec{n})a_{\vec{n}+\vec{e}_i,k} = b_{\vec{n}+\vec{e}_j,j}a_{\vec{n},k} - B_{j,k}(\vec{n})a_{\vec{n}+\vec{e}_j,k}. \quad (3.10)$$

If we take this for $k = i$ then we find

$$\frac{a_{\vec{n},i}}{a_{\vec{n}+\vec{e}_j,i}} = \frac{b_{\vec{n}-\vec{e}_i,j} - b_{\vec{n}-\vec{e}_i,i}}{b_{\vec{n}+\vec{e}_i,j} - b_{\vec{n}+\vec{e}_j,i}},$$

which gives (3.8) after using (3.6) to simplify the denominator on the right hand side. By taking $k = j$ in (3.10) we find (3.8) again but with i and j interchanged. \square

The partial difference relations can be written as

$$\begin{aligned} \Delta_i b_{\vec{n},j} &= \Delta_j b_{\vec{n},i}, \\ \Delta_j \sum_{k=1}^r a_{\vec{n},k} - \Delta_i \sum_{k=1}^r a_{\vec{n},k} &= \det \begin{pmatrix} \Delta_j b_{\vec{n},i} & b_{\vec{n},i} \\ \Delta_i b_{\vec{n},j} & b_{\vec{n},j} \end{pmatrix}, \\ \Delta_j \log a_{\vec{n},i} &= \nabla_i \log (b_{\vec{n},j} - b_{\vec{n},i}), \end{aligned}$$

4 The Christoffel-Darboux formula

Daems and Kuijlaars [10] have given the following Christoffel-Darboux formula for multiple orthogonal polynomials

Theorem 4.1. *Suppose $(\vec{n}_i)_{i=0,1,\dots,|\vec{n}|}$ is a path in \mathbb{N}^r such that $\vec{n}_0 = \vec{0}$, $\vec{n}_{|\vec{n}|} = \vec{n}$ and for every $i \in \{0, 1, \dots, |\vec{n}| - 1\}$ one has $\vec{n}_{i+1} - \vec{n}_i = \vec{e}_k$ for some $k \in \{1, 2, \dots, r\}$. Then*

$$(x - y) \sum_{i=0}^{|\vec{n}|-1} P_{\vec{n}_i}(x) Q_{\vec{n}_{i+1}}(y) = P_{\vec{n}}(x) Q_{\vec{n}}(y) - \sum_{j=1}^r a_{\vec{n},j} P_{\vec{n}-\vec{e}_j}(x) Q_{\vec{n}+\vec{e}_j}(y). \quad (4.1)$$

Observe that the right hand side is independent of the path $(\vec{n}_i)_{i=0,1,\dots,|\vec{n}|}$ in \mathbb{N}^r . A similar formule for multiple orthogonal polynomials of mixed type is given in [1]. We will give a simple proof based on the nearest neighbor recurrence relations (1.8) and the partial difference equations in Theorem 3.2.

Proof. Use the recurrence relation (1.7) for \vec{n}_i and $\vec{n}_i + \vec{e}_k = \vec{n}_{i+1}$ to find

$$x P_{\vec{n}_i}(x) = P_{\vec{n}_{i+1}}(x) + b_{\vec{n}_i,k} P_{\vec{n}_i}(x) + \sum_{j=1}^r a_{\vec{n}_i,j} P_{\vec{n}_i-\vec{e}_j}(x). \quad (4.2)$$

In a similar way we can use the recurrence relation (1.10) for \vec{n}_{i+1} and $\vec{n}_{i+1} - \vec{e}_k = \vec{n}_i$, but with x replaced by y , to find

$$y Q_{\vec{n}_{i+1}}(y) = Q_{\vec{n}_i}(y) + b_{\vec{n}_i,k} Q_{\vec{n}_{i+1}}(y) + \sum_{j=1}^r a_{\vec{n}_{i+1},j} Q_{\vec{n}_{i+1}+\vec{e}_j}(y). \quad (4.3)$$

Multiply (4.2) by $Q_{\vec{n}_{i+1}}(y)$ and (4.3) by $P_{\vec{n}_i}(x)$ and subtract both equations, to find

$$(x-y)P_{\vec{n}_i}(x)Q_{\vec{n}_{i+1}}(y) = P_{\vec{n}_{i+1}}(x)Q_{\vec{n}_{i+1}}(y) - P_{\vec{n}_i}(x)Q_{\vec{n}_i}(y) + \sum_{j=1}^r a_{\vec{n}_i,j}P_{\vec{n}_i-\vec{e}_j}(x)Q_{\vec{n}_{i+1}}(y) - \sum_{j=1}^r a_{\vec{n}_{i+1},j}P_{\vec{n}_i}(x)Q_{\vec{n}_{i+1}+\vec{e}_j}(y). \quad (4.4)$$

If we sum this for $i = 0$ to $|\vec{n}| - 1$, then the left hand side corresponds to the left hand side in (4.1), and the first terms on the right hand side give

$$\sum_{i=0}^{|\vec{n}|-1} (P_{\vec{n}_{i+1}}(x)Q_{\vec{n}_{i+1}}(y) - P_{\vec{n}_i}(x)Q_{\vec{n}_i}(y)) = P_{\vec{n}}(x)Q_{\vec{n}}(y)$$

by using the telescoping property and $Q_{\vec{0}} = 0$. The two sums on the right hand side of (4.4) need to be rewritten a bit. Observe that the recurrence relation (1.7) implies that

$$P_{\vec{n}+\vec{e}_k}(x) - P_{\vec{n}+\vec{e}_j}(x) = (b_{\vec{n},j} - b_{\vec{n},k})P_{\vec{n}}(x).$$

If we use this for $\vec{n} = \vec{n}_i - \vec{e}_j$, then this gives

$$P_{\vec{n}_i}(x) = P_{\vec{n}_{i+1}-\vec{e}_j}(x) + (b_{\vec{n}_i-\vec{e}_j,k} - b_{\vec{n}_i-\vec{e}_j,j})P_{\vec{n}_i-\vec{e}_j}(x). \quad (4.5)$$

In a similar way, the recurrence relation (1.10) implies

$$Q_{\vec{n}-\vec{e}_k}(y) - Q_{\vec{n}-\vec{e}_j}(y) = (b_{\vec{n}-\vec{e}_j,j} - b_{\vec{n}-\vec{e}_k,k})Q_{\vec{n}}(y).$$

If we use this for $\vec{n} = \vec{n}_{i+1} + \vec{e}_j$, then we find

$$Q_{\vec{n}_{i+1}}(y) = Q_{\vec{n}_i+\vec{e}_j}(y) + (b_{\vec{n}_i+\vec{e}_j,k} - b_{\vec{n}_{i+1},j})Q_{\vec{n}_{i+1}+\vec{e}_j}(y). \quad (4.6)$$

Replace $P_{\vec{n}_i}(x)$ in the second sum of (4.4) by (4.5) and replace $Q_{\vec{n}_{i+1}}(y)$ in the first sum of (4.4) by (4.6), then we find that

$$\begin{aligned} & \sum_{j=1}^r a_{\vec{n}_i,j}P_{\vec{n}_i-\vec{e}_j}(x)Q_{\vec{n}_{i+1}}(y) - \sum_{j=1}^r a_{\vec{n}_{i+1},j}P_{\vec{n}_i}(x)Q_{\vec{n}_{i+1}+\vec{e}_j}(y) \\ &= \sum_{j=1}^r a_{\vec{n}_i,j}P_{\vec{n}_i-\vec{e}_j}(x)Q_{\vec{n}_i+\vec{e}_j}(y) - \sum_{j=1}^r a_{\vec{n}_{i+1},j}P_{\vec{n}_{i+1}-\vec{e}_j}(x)Q_{\vec{n}_{i+1}+\vec{e}_j}(y) \\ &+ \sum_{j=1}^r P_{\vec{n}_i-\vec{e}_j}(x)Q_{\vec{n}_{i+1}+\vec{e}_j}(y) [a_{\vec{n}_i,j}(b_{\vec{n}_i+\vec{e}_j,k} - b_{\vec{n}_{i+1},j}) - a_{\vec{n}_{i+1},j}(b_{\vec{n}_i-\vec{e}_j,k} - b_{\vec{n}_i-\vec{e}_j,j})]. \end{aligned}$$

The last sum on the right hand side vanishes because of the partial difference relations in Theorem 3.2, in particular (3.8) and (3.6) but with $i = k$. The other two sums are now of the form $S_i - S_{i+1}$, so that we are summing a telescoping series, giving

$$\sum_{i=0}^{|\vec{n}|-1} (S_i - S_{i+1}) = S_0 - S_{|\vec{n}|} = - \sum_{j=1}^r a_{\vec{n},j}P_{\vec{n}-\vec{e}_j}(x)Q_{\vec{n}+\vec{e}_j}(y),$$

which is the sum on the right hand side of (4.1). \square

5 Some examples

In this section we give several examples of multiple orthogonal polynomials and their nearest neighbor recurrence coefficients. For each example one can verify that the partial difference equations in Theorem 3.2 indeed hold. Some of these examples were given before, e.g., in [3], [4], [5], [7], [9], [15, Chap. 23], [20], but the explicit expression for the recurrence coefficients appears here for the first time for most of these families.

5.1 Multiple Hermite polynomials

See [20, §3.4], [15, §23.5], [7]. These are given by

$$\int_{-\infty}^{\infty} x^k H_{\vec{n}}(x) e^{-x^2 + c_j x} dx = 0, \quad k = 0, 1, \dots, n_j - 1,$$

for $1 \leq j \leq r$, where $c_i \neq c_j$ whenever $i \neq j$. The recurrence relation is explicitly given as

$$x H_{\vec{n}}(x) = H_{\vec{n} + \vec{e}_k}(x) + \frac{c_k}{2} H_{\vec{n}}(x) + \frac{1}{2} \sum_{j=1}^r n_j H_{\vec{n} - \vec{e}_j}(x),$$

for $1 \leq k \leq r$, so that

$$b_{\vec{n},j} = c_j/2, \quad a_{\vec{n},j} = n_j/2, \quad 1 \leq j \leq r.$$

5.2 Multiple Charlier polynomials

See [4, §4.1], [15, §23.6.1]. The orthogonality relations are

$$\sum_{k=0}^{\infty} C_{\vec{n}}(k) k^{\ell} \frac{a_j^k}{k!} = 0, \quad \ell = 0, 1, \dots, n_j - 1,$$

for $1 \leq j \leq r$, where $a_i > 0$ and $a_i \neq a_j$ whenever $i \neq j$. The recurrence relation is given by

$$x C_{\vec{n}}(x) = C_{\vec{n} + \vec{e}_k}(x) + (a_k + |\vec{n}|) C_{\vec{n}}(x) + \sum_{j=1}^r n_j a_j C_{\vec{n} - \vec{e}_j}(x),$$

so that

$$b_{\vec{n},j} = |\vec{n}| + a_j, \quad a_{\vec{n},j} = n_j a_j, \quad 1 \leq j \leq r.$$

5.3 Multiple Laguerre polynomials of the first kind

See [20, §3.2] and [15, §23.4.1], [7]. These are given by the orthogonality relations

$$\int_0^{\infty} x^k L_{\vec{n}}(x) x^{\alpha_j} e^{-x} dx = 0, \quad k = 0, 1, \dots, n_j - 1,$$

for $1 \leq j \leq r$, where $\alpha_1, \dots, \alpha_r > -1$ and $\alpha_i - \alpha_j \notin \mathbb{Z}$. They can be obtained using the Rodrigues formula

$$(-1)^{|\vec{n}|} e^{-x} L_{\vec{n}}(x) = \prod_{j=1}^r \left(x^{-\alpha_j} \frac{d^{n_j}}{dx^{n_j}} x^{n_j + \alpha_j} \right) e^{-x} \quad (5.1)$$

where the product of the differential operators can be taken in any order. This Rodrigues formula is useful for computing the recurrence coefficients. Indeed, we easily find

$$\int_0^\infty x^{n_j} L_{\vec{n}}(x) x^{\alpha_j} e^{-x} = (-1)^{|\vec{n}|} \int_0^\infty x^{n_j + \alpha_j - \alpha_1} \frac{d^{n_1}}{dx^{n_1}} x^{n_1 + \alpha_1} \prod_{i=2}^r \left(x^{-\alpha_i} \frac{d^{n_i}}{dx^{n_i}} x^{n_i + \alpha_i} \right) e^{-x} dx$$

and integration by parts (n_1 times) gives

$$\begin{aligned} &= (-1)^{|\vec{n}| + n_1} \int_0^\infty \left(\frac{d^{n_1}}{dx^{n_1}} x^{n_j + \alpha_j - \alpha_1} \right) x^{n_1 + \alpha_1} \prod_{i=2}^r \left(x^{-\alpha_i} \frac{d^{n_i}}{dx^{n_i}} x^{n_i + \alpha_i} \right) e^{-x} dx \\ &= (-1)^{|\vec{n}| + n_1} \binom{n_j + \alpha_j - \alpha_1}{n_1} n_1! \int_0^\infty x^{n_j + \alpha_j} \prod_{i=2}^r \left(x^{-\alpha_i} \frac{d^{n_i}}{dx^{n_i}} x^{n_i + \alpha_i} \right) e^{-x} dx. \end{aligned}$$

Repeating this r times gives

$$\begin{aligned} \int_0^\infty x^{n_j} L_{\vec{n}}(x) x^{\alpha_j} e^{-x} &= \prod_{i=1}^r \binom{n_j + \alpha_j - \alpha_i}{n_i} n_i! \int_0^\infty x^{n_j + \alpha_j} e^{-x} dx \\ &= \Gamma(n_j + \alpha_j + 1) \prod_{i=1}^r \binom{n_j + \alpha_j - \alpha_i}{n_i} n_i!. \end{aligned}$$

If we then use (1.8) then we find

$$a_{\vec{n},j} = n_j(n_j + \alpha_j) \prod_{i=1, i \neq j}^r \frac{n_j + \alpha_j - \alpha_i}{n_j - n_i + \alpha_j - \alpha_i}, \quad j = 1, \dots, r. \quad (5.2)$$

The recurrence coefficients $b_{\vec{n},k}$ can be obtained by comparing the coefficients of $x^{|\vec{n}|}$ in the recurrence relation (1.7). They are

$$b_{\vec{n},k} = |\vec{n}| + n_k + \alpha_k + 1, \quad k = 1, \dots, r. \quad (5.3)$$

These recurrence coefficients are here given for the first time. Observe that for $r = 1$ one retrieves the well known recurrence coefficients for the monic Laguerre polynomials $a_n^2 = n(n + \alpha)$ and $b_n = 2n + \alpha + 1$.

An interesting observation is that these recurrence coefficients of multiple Laguerre polynomials of the first kind satisfy

$$\sum_{j=1}^r a_{\vec{n},j} = \sum_{j=1}^r n_j \alpha_j + \frac{1}{2} \left(|\vec{n}|^2 + \sum_{j=1}^r n_j^2 \right) > 0. \quad (5.4)$$

Indeed, if we introduce the polynomials

$$q_r(x) = \prod_{j=1}^r (x - \alpha_j), \quad Q_{r,\vec{n}}(x) = \prod_{j=1}^r (x - n_j - \alpha_j),$$

then $a_{\vec{n},j} = (n_j + \alpha_j) q_r(n_j + \alpha_j) / Q'_{r,\vec{n}}(n_j + \alpha_j)$ and hence by the residue theorem

$$\sum_{j=1}^r a_{\vec{n},j} = \frac{1}{2\pi i} \int_{\Gamma} \frac{z q_r(z)}{Q_{r,\vec{n}}(z)} dz$$

where Γ is a closed contour around all the zeros $n_i + \alpha_i$ ($i = 1, \dots, r$) of $Q_{r,\vec{n}}$. We can take for instance the circle $C_R = \{Re^{i\theta} : \theta \in [0, 2\pi]\}$ for R large enough. If we change the variable $z = 1/\xi$, then the integral becomes

$$\frac{1}{2\pi i} \int_{C_r} \frac{q_r^*(\xi)}{\xi^3 Q_{r,\vec{n}}^*(\xi)} d\xi$$

where C_r is the circle with radius $r = 1/R$ and the orientation is positive by changing the sign appropriately. Here $q_r^*(z) = z^r q_r(1/z)$ and $Q_{r,\vec{n}}^*(z) = z^r Q_{r,\vec{n}}(1/z)$ are the polynomials taking in reversed order. The residue theorem now tells us that the integral is the residue of $q_r^*(\xi)/\xi^3 Q_{r,\vec{n}}^*(\xi)$ at $\xi = 0$, and this can easily be computed by using

$$\frac{1 - \xi \alpha_j}{1 - \xi(n_j + \alpha_j)} = 1 + \xi n_j + \xi^2(n_j + \alpha_j) + \mathcal{O}(\xi^3),$$

so that

$$\frac{q_r^*(\xi)}{Q_{r,\vec{n}}^*(\xi)} = 1 + \xi |\vec{n}| + \xi^2 \left(\sum_{j=1}^r n_j(n_j + \alpha_j) + \sum_{i < j} n_i n_j \right) + \mathcal{O}(\xi^3),$$

and the residue of $q_r^*(\xi)/\xi^3 Q_{r,\vec{n}}^*(\xi)$ thus corresponds to the coefficient of the quadratic term, giving (5.4).

5.4 Multiple Laguerre polynomials of the second kind

See [17, Remark 5 on p. 160], [20, §3.3] and [15, §23.4.2], [7]. These are given by the orthogonality relations

$$\int_0^\infty x^k L_{\vec{n}}(x) x^\alpha e^{-c_j x} dx = 0, \quad k = 0, 1, \dots, n_j - 1,$$

for $1 \leq j \leq r$, where $\alpha > -1$, $c_1, \dots, c_r > 0$ and $c_i \neq c_j$ whenever $i \neq j$. They can be obtained using the Rodrigues formula

$$(-1)^{|\vec{n}|} \left(\prod_{j=1}^r c_j^{n_j} \right) x^\alpha L_{\vec{n}}(x) = \prod_{j=1}^r \left(e^{c_j x} \frac{d^{n_j}}{dx^{n_j}} e^{-c_j x} \right) x^{|\vec{n}|+\alpha} \quad (5.5)$$

where the differential operators in the product can be taken in any order. A useful integral is

$$\int_0^\infty e^{-\lambda x} \prod_{j=1}^r \left(e^{c_j x} \frac{d^{n_j}}{dx^{n_j}} e^{-c_j x} \right) x^{|\vec{n}|+\alpha} dx = (-1)^{|\vec{n}|} \frac{\Gamma(|\vec{n}| + \alpha + 1)}{\lambda^{|\vec{n}|+\alpha+1}} \prod_{j=1}^r (c_j - \lambda)^{n_j}$$

which can be evaluated by using integration by parts in a similar way as in the previous example. Observe that the right hand side has a zero at $\lambda = c_j$ of multiplicity n_j . Using (5.5) we thus have for $\lambda > 0$

$$\int_0^\infty e^{-\lambda x} x^\alpha L_{\vec{n}}(x) dx = \frac{\Gamma(|\vec{n}| + \alpha + 1)}{\lambda^{|\vec{n}|+\alpha+1}} \prod_{i=1}^r (1 - \lambda/c_i)^{n_i}.$$

Clearly

$$\left. \frac{d^k}{d\lambda^k} \int_0^\infty e^{-\lambda x} x^\alpha L_{\vec{n}}(x) dx \right|_{\lambda=c_j} = (-1)^k \int_0^\infty x^k e^{c_j x} x^\alpha L_{\vec{n}}(x) dx = 0, \quad 0 \leq k < n_j,$$

which confirms the orthogonality relations, and for $k = n_j$

$$\int_0^\infty x^{n_j} e^{c_j x} x^\alpha L_{\vec{n}}(x) dx = \frac{\Gamma(|\vec{n}| + \alpha + 1)}{c_j^{|\vec{n}| + n_j + \alpha + 1}} n_j! \prod_{i=1, i \neq j}^r \left(1 - \frac{c_j}{c_i}\right).$$

If we then use (1.8) then we find

$$a_{\vec{n},j} = \frac{(|\vec{n}| + \alpha) n_j}{c_j^2}, \quad 1 \leq j \leq r. \quad (5.6)$$

For the coefficients $b_{\vec{n},k}$ we compare the coefficients of $x^{|\vec{n}|}$ on both sides of the recurrence relation (1.7) and use the explicit expression (23.4.5) in [15] to find

$$b_{\vec{n},k} = \frac{|\vec{n}| + \alpha + 1}{c_k} + \sum_{j=1}^r \frac{n_j}{c_j}. \quad (5.7)$$

Observe that for $r = 1$ and $c_1 = 1$ we retrieve the recurrence coefficients of monic Laguerre polynomials $b_n = 2n + \alpha + 1$ and $a_n^2 = n(n + \alpha)$.

5.5 Jacobi-Piñeiro polynomials

See [17, Remark 7 on p. 162], [20, §2.1] and [15, §23.3.2]. These are multiple orthogonal polynomials on $[0, 1]$ for the Jacobi weights $x_i^{\alpha_i} (1 - x)^\beta$, with $\alpha_1, \dots, \alpha_r, \beta > -1$ and $\alpha_i - \alpha_j \notin \mathbb{Z}$. They satisfy

$$\int_0^1 P_{\vec{n}}(x) x^k x^{\alpha_j} (1 - x)^\beta dx = 0, \quad k = 0, 1, \dots, n_j - 1, \quad 1 \leq j \leq r.$$

They are given by the Rodrigues formula

$$\begin{aligned} (-1)^{|\vec{n}|} \prod_{j=1}^r (|\vec{n}| + \alpha_j + \beta + 1)_{n_j} (1 - x)^\beta P_{\vec{n}}(x) \\ = \prod_{j=1}^r \left(x^{-\alpha_j} \frac{d^{n_j}}{dx^{n_j}} x^{n_j + \alpha_j} \right) (1 - x)^{|\vec{n}| + \beta}, \end{aligned} \quad (5.8)$$

where the product of differential operators is the same as for the multiple Laguerre polynomials of the first kind. One has

$$\int_0^1 x^\gamma P_{\vec{n}}(x) (1 - x)^\beta dx = (-1)^{|\vec{n}|} \frac{\prod_{i=1}^r (\alpha_i - \gamma)_{n_i}}{\prod_{i=1}^r (|\vec{n}| + \alpha_i + \beta + 1)_{n_i}} \frac{\Gamma(\gamma + 1) \Gamma(|\vec{n}| + \beta + 1)}{\Gamma(|\vec{n}| + \beta + \gamma + 2)}$$

(see [15, §23/3.2] which has an extra $(-1)^{|\vec{n}|}$ that shouldn't be there) so that

$$\int_0^1 x^{n_j + \alpha_j} P_{\vec{n}}(x) (1 - x)^\beta dx = \frac{\prod_{i=1}^r \binom{n_j + \alpha_j - \alpha_i}{n_i} n_i!}{\prod_{i=1}^r (|\vec{n}| + \alpha_i + \beta + 1)_{n_i}} \frac{\Gamma(n_j + \alpha_j + 1) \Gamma(|\vec{n}| + \beta + 1)}{\Gamma(|\vec{n}| + n_j + \alpha_j + \beta + 2)}.$$

If we use (1.8) we then find for $1 \leq j \leq r$

$$a_{\vec{n},j} = \prod_{i=1, i \neq j}^r \frac{n_j + \alpha_j - \alpha_i}{n_j - n_i + \alpha_j - \alpha_i} \prod_{i=1}^r \frac{|\vec{n}| + \alpha_i + \beta}{|\vec{n}| + n_i + \alpha_i + \beta} \\ \times \frac{n_j(n_j + \alpha_j)(|\vec{n}| + \beta)}{(|\vec{n}| + n_j + \alpha_j + \beta + 1)(|\vec{n}| + n_j + \alpha_j + \beta)(|\vec{n}| + n_j + \alpha_j + \beta - 1)}. \quad (5.9)$$

We can write this as

$$a_{\vec{n},j} = \frac{q_r(-|\vec{n}| - \beta)}{Q_{r,\vec{n}}(-|\vec{n}| - \beta)} \frac{q_r(n_j + \alpha_j)}{Q'_{r,\vec{n}}(n_j + \alpha_j)} \\ \times \frac{(n_j + \alpha_j)(|\vec{n}| + \beta)}{(|\vec{n}| + n_j + \alpha_j + \beta + 1)(|\vec{n}| + n_j + \alpha_j + \beta)(|\vec{n}| + n_j + \alpha_j + \beta - 1)},$$

where the polynomials q_r and $Q_{r,\vec{n}}$ are given, as before, by

$$q_r(x) = \prod_{j=1}^r (x - \alpha_j), \quad Q_{r,\vec{n}}(x) = \prod_{j=1}^r (x - n_j - \alpha_j). \quad (5.10)$$

Observe that for $r = 1$ we retrieve the recurrence coefficients

$$a_n^2 = \frac{n(n + \alpha)(n + \beta)(n + \alpha + \beta)}{(2n + \alpha + \beta - 1)(2n + \alpha + \beta)^2(2n + \alpha + \beta + 1)}$$

for monic Jacobi polynomials on $[0, 1]$.

The other recurrence coefficients can be obtained by $b_{\vec{n},k} = \delta_{\vec{n}} - \delta_{\vec{n} + \vec{e}_k}$, where $\delta_{\vec{n}}$ is the coefficient of $x^{|\vec{n}|-1}$ of the polynomial $P_{\vec{n}}(x)$:

$$P_{\vec{n}}(x) = x^{|\vec{n}|} + \delta_{\vec{n}} x^{|\vec{n}|-1} + \dots$$

For Jacobi-Piñeiro polynomials we have

$$\delta_{\vec{n}} = -(|\vec{n}| + \beta) \frac{q_r(-|\vec{n}| - \beta)}{Q_{r,\vec{n}}(-|\vec{n}| - \beta)} + \beta, \quad (5.11)$$

where the polynomials q_r and $Q_{r,\vec{n}}$ are given in (5.10). To see that this is true, one can use the Rodrigues formula (5.8) to find that

$$\left(x^{\alpha_i+1} (1-x)^{\beta+1} P_{\vec{n}-\vec{e}_i}^{(\vec{\alpha}+\vec{e}_i, \beta+1)}(x) \right)' = -(|\vec{n}| + \alpha_i + \beta + 1) x^{\alpha_i} (1-x)^{\beta} P_{\vec{n}}^{(\vec{\alpha}, \beta)}(x)$$

(see also [9, Eq. (1.4)]). If we identify the coefficients in this identity, then

$$(|\vec{n}| + \alpha_i + \beta + 1) \delta_{\vec{n}}(\vec{\alpha}, \beta) = (|\vec{n}| + \alpha_i + \beta) \delta_{\vec{n}-\vec{e}_i}(\vec{\alpha} + \vec{e}_i, \beta + 1) - |\vec{n}| - \alpha_i.$$

This can be used to prove (5.11) by induction on r . Observe that we can use a decomposition in partial fractions to write

$$\frac{q_r(z)}{Q_{r,\vec{n}}(z)} = 1 + \sum_{j=1}^r \frac{q_r(n_j + \alpha_j)/Q'_{r,\vec{n}}(n_j + \alpha_j)}{z - n_j - \alpha_j},$$

so that

$$\frac{q_r(-|\vec{n}| - \beta)}{Q_{r,\vec{n}}(-|\vec{n}| - \beta)} = 1 - \sum_{j=1}^r \frac{q_r(n_j + \alpha_j)/Q'_{r,\vec{n}}(n_j + \alpha_j)}{|\vec{n}| + n_j + \alpha_j + \beta}$$

and

$$\delta_{\vec{n}} = -|\vec{n}| + (|\vec{n}| + \beta) \sum_{j=1}^r \frac{q_r(n_j + \alpha_j)/Q'_{r,\vec{n}}(n_j + \alpha_j)}{|\vec{n}| + n_j + \alpha_j + \beta}.$$

Furthermore

$$\sum_{j=1}^r q_r(n_j + \alpha_j)/Q'_{r,\vec{n}}(n_j + \alpha_j) = |\vec{n}|$$

so that a little calculus gives

$$\delta_{\vec{n}} = - \sum_{j=1}^r \frac{(n_j + \alpha_j) q_r(n_j + \alpha_j)/Q'_{r,\vec{n}}(n_j + \alpha_j)}{|\vec{n}| + n_j + \alpha_j + \beta}.$$

Note that for $r = 1$ this indeed gives

$$\delta_n = \frac{-n(n + \alpha)}{2n + \alpha + \beta},$$

from which

$$b_n = \delta_n - \delta_{n+1} = \frac{1}{2} + \frac{\beta^2 - \alpha^2}{2(2n + \alpha + \beta)(2n + \alpha + \beta + 2)},$$

which is indeed the recurrence coefficient b_n for the Jacobi polynomials on $[0, 1]$.

We can find a nice closed form for the sum $\sum_{j=1}^r a_{\vec{n},j}$. We have

$$\begin{aligned} \sum_{j=1}^r a_{\vec{n},j} &= (|\vec{n}| + \beta) \frac{q_r(-|\vec{n}| - \beta)}{Q_{r,\vec{n}}(-|\vec{n}| - \beta)} \\ &\times \sum_{j=1}^r \frac{q_r(n_j + \alpha_j)}{Q'_{r,\vec{n}}(n_j + \alpha_j)} \frac{n_j + \alpha_j}{(|\vec{n}| + n_j + \alpha_j + \beta + 1)(|\vec{n}| + n_j + \alpha_j + \beta)(|\vec{n}| + n_j + \alpha_j + \beta - 1)}, \end{aligned}$$

and for the sum in this expression, we can use the residue theorem to find

$$\begin{aligned} \sum_{j=1}^r \frac{q_r(n_j + \alpha_j)}{Q'_{r,\vec{n}}(n_j + \alpha_j)} \frac{n_j + \alpha_j}{(|\vec{n}| + n_j + \alpha_j + \beta + 1)(|\vec{n}| + n_j + \alpha_j + \beta)(|\vec{n}| + n_j + \alpha_j + \beta - 1)} \\ = \frac{1}{2\pi i} \int_{\Gamma} \frac{q_r(z)}{Q_{r,\vec{n}}(z)} \frac{z dz}{(z + |\vec{n}| + \beta + 1)(z + |\vec{n}| + \beta)(z + |\vec{n}| + \beta - 1)}, \end{aligned}$$

where Γ is a closed contour around all the zeros $n_i + \alpha_i$ ($i = 1, \dots, r$) of $Q_{r,\vec{n}}$ but with $-|\vec{n}| - \beta$, $-|\vec{n}| - \beta \pm 1$ outside the contour. If we choose Γ so that 0 is inside and change the variable $z = 1/\xi$, then the integral becomes

$$\frac{1}{2\pi i} \int_{\Gamma^*} \frac{q_r^*(\xi)}{Q_{r,\vec{n}}^*(\xi)} \frac{d\xi}{(1 + \xi(|\vec{n}| + \beta + 1))(1 + \xi(|\vec{n}| + \beta))(1 + \xi(|\vec{n}| + \beta - 1))},$$

where Γ^* is now a contour around 0 with the poles $-1/(|\vec{n} + \beta)$ and $-1/(|\vec{n}| + \beta \pm 1)$ inside and the zeros of $Q_{r,\vec{n}}^*$ outside. The polynomials q_r^* and $Q_{r,\vec{n}}^*$ are the reversed polynomials (as in §5.3). The residue theorem therefore evaluates this contour integral as

$$\frac{1}{2} \frac{q_r(-|\vec{n}| - \beta - 1)}{Q_{r,\vec{n}}(-|\vec{n}| - \beta - 1)} (|\vec{n}| + \beta - 1) - \frac{q_r(-|\vec{n}| - \beta)}{Q_{r,\vec{n}}(-|\vec{n}| - \beta)} (|\vec{n}| + \beta) + \frac{1}{2} \frac{q_r(-|\vec{n}| - \beta + 1)}{Q_{r,\vec{n}}(-|\vec{n}| - \beta + 1)} (|\vec{n}| + \beta + 1),$$

and we conclude that

$$\begin{aligned} \sum_{j=1}^r a_{\vec{n},j} = (|\vec{n}| + \beta) & \frac{q_r(-|\vec{n}| - \beta)}{Q_{r,\vec{n}}(-|\vec{n}| - \beta)} \left(\frac{1}{2} \frac{q_r(-|\vec{n}| - \beta - 1)}{Q_{r,\vec{n}}(-|\vec{n}| - \beta - 1)} (|\vec{n}| + \beta - 1) \right. \\ & \left. - \frac{q_r(-|\vec{n}| - \beta)}{Q_{r,\vec{n}}(-|\vec{n}| - \beta)} (|\vec{n}| + \beta) + \frac{1}{2} \frac{q_r(-|\vec{n}| - \beta + 1)}{Q_{r,\vec{n}}(-|\vec{n}| - \beta + 1)} (|\vec{n}| + \beta + 1) \right). \end{aligned}$$

Observe that the expression between brackets is $-1/2$ times the second difference $\Delta \nabla$ of $z q_r(z)/Q_{r,\vec{n}}(z)$ evaluated at $z = -|\vec{n}| - \beta$ and that this expression vanishes when $|\vec{n}| = 0$.

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